Grupa A, Pismeni ispit iz Matematike II, 04.07.2013.
ispit pisati isključivo hemiskom olovkom

1. Figura u ravni ograničena linijama $2y = x^2$ i $2x + 2y - 3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

2. Izračunati $\iint\limits_D y \mathrm{d}x \mathrm{d}y$ gdje je $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 2x,\ y \geq 0\}$.

3. Izračunati

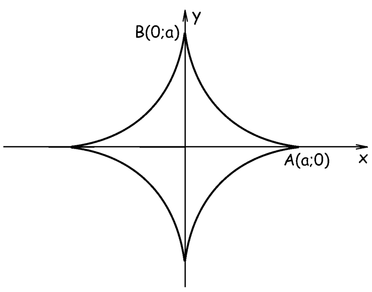
$$I = \int\limits_C (e^{x+y} \sin 2y + x + y)\mathrm{d}x + (e^{x+y}(2 \cos 2y + \sin 2y) + 2x)\mathrm{d}y$$

gdje je C kriva $y = \sqrt{2x - x^2}$, integracija se vrši od tačke $A(2; 0)$ do tačke $O(0; 0)$.

4. Prvo izračunati integral $I = \int\limits_0^\infty e^{-x} \sin(\alpha x)\mathrm{d}x$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha) = \int\limits_0^\infty x e^{-x} \cos(\alpha x)\mathrm{d}x$$

Grupa B, Pismeni ispit iz Matematike II, 04.07.2013.
ispit pisati isključivo hemiskom olovkom



1. Izračunati zapreminu tijela koje nastaje rotacijom krive $x = a\cos^3 t$, $y = a \sin^3 t$ oko x-ose (data kriva je poznata pod imenom astroida i njen grafik je prikazan na slici lijevo).

2. Izračunati $\iint\limits_D x \mathrm{d}x \mathrm{d}y$ gdje je $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 2y,\ x \leq y,\ x \geq 0\}$.

3. Izračunati

$$I = \int\limits_{\overset{\frown}{AO}} (e^x \sin y - my)\mathrm{d}x + (e^x \cos y - m)\mathrm{d}y$$

gdje je $\overset{\frown}{AO}$ gornji polukrug $x^2 + y^2 = ax$, $y \geq 0$ $(a > 0)$ orjentisan od tačke $A(a; 0)$ do tačke $O(0; 0)$.

4. Date su vrijednosti dva integrala $(\alpha > 0)$

$$\int_0^\infty \frac{\cos \alpha x}{1 + x^2}\mathrm{d}x = \frac{\pi}{2}e^{-\alpha}, \quad \int_0^\infty \frac{\sin \alpha x}{x}\mathrm{d}x = \frac{\pi}{2}.$$

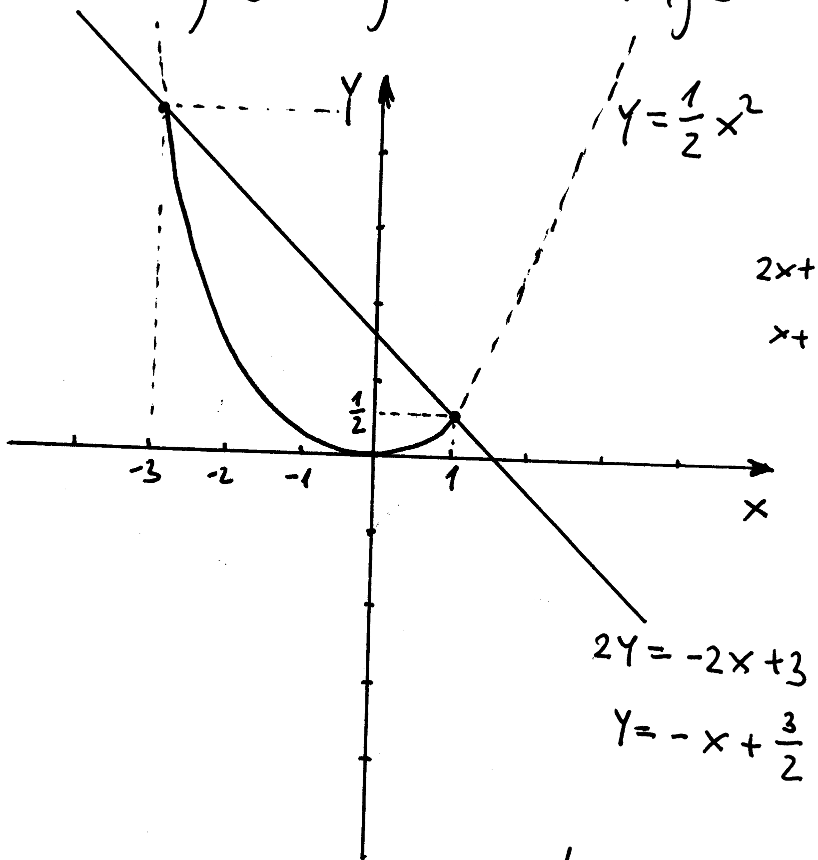
Koristeći date jednakosti, uz pomoć metode diferenciranja po parametru izračunati $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x(1 + x^2)}\mathrm{d}x$.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

(#) Figura u ravni ograničena linijama $2y = x^2$ i $2x + 2y - 3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj. Skicirajmo dvije date linije

$$y = \frac{1}{2}x^2$$



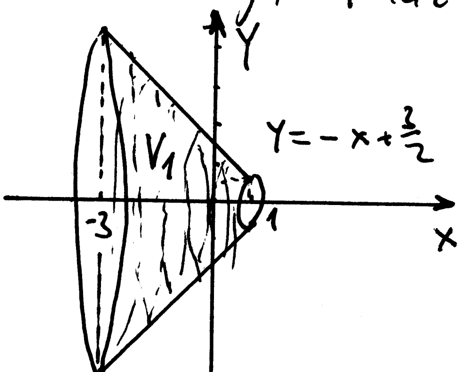
$$2x + 2y - 3 = 0 \;/:2$$
$$x + y - \frac{3}{2} = 0$$

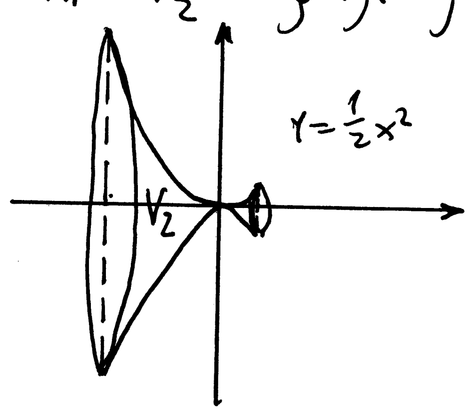
$$2y = x^2$$
$$2x + 2y - 3 = 0$$

$$x^2 + 2x - 3 = 0$$
$$(x+3)(x-1) = 0$$
$$x_1 = -3 \Rightarrow y_1 = \frac{9}{2}$$
$$x_2 = 1 \Rightarrow y_2 = \frac{1}{2}$$

Prisjetimo se: $V_x = \pi \int_a^b y^2\, dx$ je zapremina tijela kada f-ja $y = f(x)$ rotira oko x-ose, za $a \le x \le b$

U našem slučaju imademo $V = V_1 - V_2$ gdje je





$$V_1 = \pi \int_{-3}^{1} \left(-x + \frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} \left(x^2 - 3x + \frac{9}{4}\right) dx =$$

$$= \frac{1}{3}x^3 \Big|_{-3}^{1} - \frac{3}{2}x^2 \Big|_{-3}^{1} + \frac{9}{4}x \Big|_{-3}^{1} = \dots = \frac{91}{3}\pi$$

integral u V_1 smo mogli izračunati i na drugi način

$$V_1 = \pi \int_{-3}^{1} \left(-x + \frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} (-1)^2 \left(x - \frac{3}{2}\right)^2 d\left(x - \frac{3}{2}\right) =$$

$$= \pi \cdot \frac{1}{3}\left(x - \frac{3}{2}\right)^3 \Big|_{-3}^{1} = \dots = \frac{91\pi}{3}$$

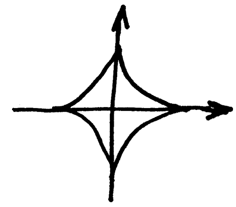
$$V_2 = \pi \int_{-3}^{1} \left(\frac{1}{2}x^2\right)^2 dx = \frac{\pi}{4} \int_{-3}^{1} x^4 dx = \frac{\pi}{4} \cdot \frac{1}{5} x^5 \Big|_{-3}^{1} = \frac{\pi}{20}(1 + 243) =$$

$$= \frac{61}{5}\pi$$

$$V = V_1 - V_2 = \frac{91\pi}{3} - \frac{61\pi}{5} = \frac{272\pi}{15} = 18\frac{2}{15}\pi$$

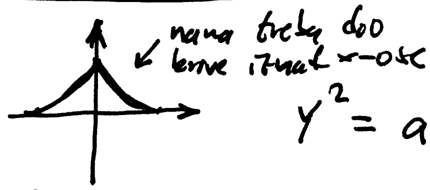
tražena zapremina

(#) Izračunati zapreminu tijela koje nastaje rotacijom krive $x=a\cos^3 t$, $y=a\sin^3 t$ oko x-ose.

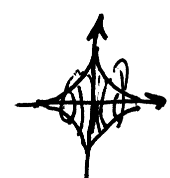
Rj. Data kriva $c:\begin{cases} x=a\cos^3 t\\ y=a\sin^3 t\\ t\in(0,2\pi]\end{cases}$ je poznata pod imenom astroida i njen grafik je  .

Prisjetimo se:

$$V_x=\pi\int_{t_1}^{t_2}[\mu(t)]^2\,|\eta'(t)|\,dt$$ zapremina tijela kada kriva c rotira oko x-ose, gdje je $c:\begin{cases} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{cases}$



$x=a\cos^3 t$ $\qquad$ $y^2=a^2\sin^6 t$



$x'=3a\cos^2 t\cdot(-\sin t)=-3a\sin t\cos^2 t$

$$V_x=\pi\int_0^{\pi}a^2\sin^6 t\cdot 3a\,|\sin t|\,|\cos^2 t|\,dt=$$ | kako je figura simetrična u odnosu na y-osu i $\sin t>0$ za $t\in(0,\frac{\pi}{2})$ |

$$=2\pi\int_0^{\pi/2}a^2\sin^6 t\cdot 3a\cdot\sin t\cdot\cos^2 t\,dt=6a^3\pi\int_0^{\pi/2}\sin^7 t\cos^2 t\,dt$$

$$=6a^3\pi\int_0^{\pi/2}\underbrace{\sin^6 t}_{(\sin^2 t)^3}\cos^2 t\sin t\,dt=6a^3\pi\int_0^{\pi/2}\underbrace{(1-\cos^2 t)^3}_{1-3\cos^2 t+3\cos^4 t-\cos^6 t}\cos^2 t\,(-1)\,d\cos t=$$

$$=-6a^2\pi\int_0^{\pi/2}(\cos^2 t-3\cos^4 t+3\cos^6 t-\cos^8 t)\,d\cos t\overset{\text{ZA VJEŽBU}}{=}\frac{32}{105}\pi a^3$$

(#) Izračunati $\iint\limits_D y\,dx\,dy$ gdje je

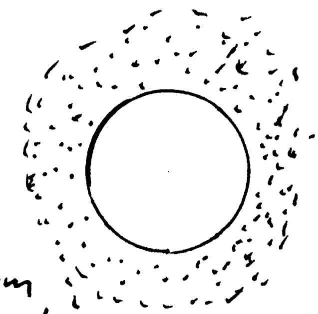
$$D=\{(x,y):\ 1\le x^2+y^2\le 2x,\ y\ge 0\}.$$

Rj:

$1\le x^2+y^2$

$x^2+y^2=1$

krug sa centrom u tački $C(0;0)$ polupr. $r=1$

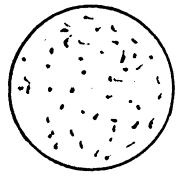


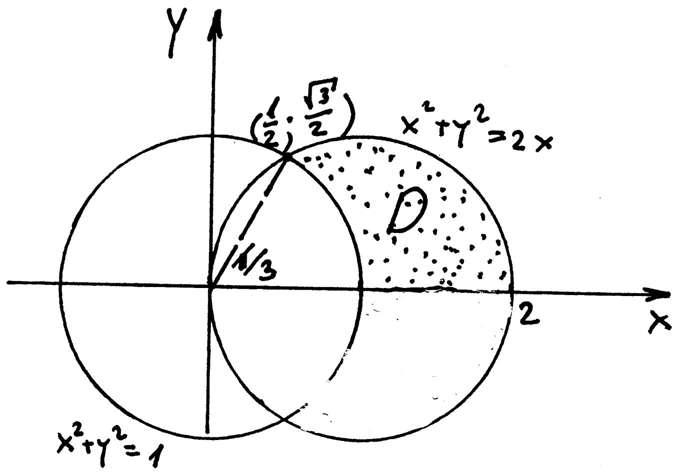
$x^2+y^2\le 2x$

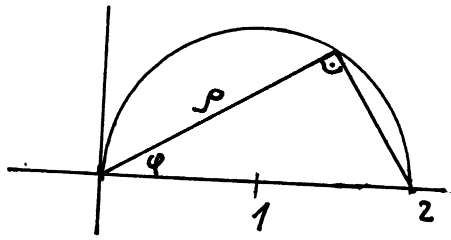
$x^2+y^2=2x$

$x^2-2\cdot x\cdot 1+1+y^2=1$

$(x-1)^2+y^2=1$







$\cos\varphi=\frac{\rho}{2}$

$\rho=2\cos\varphi$

Uvedimo polarne koordinate

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$dx\,dy=\rho\,d\rho\,d\varphi$

$D\xrightarrow{\text{transform.}}D'$

$$D':\begin{cases}0\le\varphi\le\frac{\pi}{3}\\ 1\le\rho\le 2\cos\varphi\end{cases}$$

$$\iint\limits_D y\,dx\,dy=\left|\begin{matrix}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{matrix}\right|=\iint\limits_{D'}\rho\sin\varphi\,\rho\,d\rho\,d\varphi=$$

$$=\int_0^{\frac{\pi}{3}}\sin\varphi\,d\varphi\int_1^{2\cos\varphi}\rho^2\,d\rho=\frac{1}{3}\int_0^{\pi/3}\sin\varphi\,\underbrace{\rho^3\Big|_1^{2\cos\varphi}}_{8\cos^3\varphi-1}d\varphi=\frac{8}{3}\int_0^{\pi/3}\underbrace{\sin\varphi\cos^3\varphi\,d\varphi}_{-\cos^3\varphi\,d(\cos\varphi)}-\frac{1}{3}\int_0^{\pi/3}\sin\varphi\,d\varphi$$

$$=\frac{-\cancel{8}^{\,2}}{3}\cdot\frac{1}{\cancel{4}_{\,1}}\underbrace{\cos^4\varphi\Big|_0^{\frac{\pi}{3}}}_{\left(\frac{1}{2}\right)^4-1}+\frac{1}{3}\underbrace{\cos\varphi\Big|_0^{\frac{\pi}{3}}}_{\frac{1}{2}-1}=\frac{-\cancel{2}^{\,1}}{\cancel{3}_{\,1}}\cdot\frac{\cancel{1-16}^{\,-\cancel{15}\,5}}{\cancel{16}_{\,8}}+\frac{1}{3}\cdot\frac{-1}{2}=\frac{5}{8}-\frac{1}{6}=\frac{11}{24}$$ traženo rješenje

(#) Izračunati $I = \iint\limits_D x\,dx\,dy$ gdje je

$D = \{(x,y) : 1 \le x^2+y^2 \le 4y \ \wedge \ x \le y \ \wedge \ x \ge 0\}$.

Rj.

$1 \le x^2+y^2$

$x^2+y^2=1$

krug sa centrom $C(0,0)$ poluprečnika 1

$x^2+y^2 \le 4y$

$x^2+y^2=4y$

$x^2+y^2-2\cdot y\cdot 2+4=4$

$x^2+(y-2)^2=4$

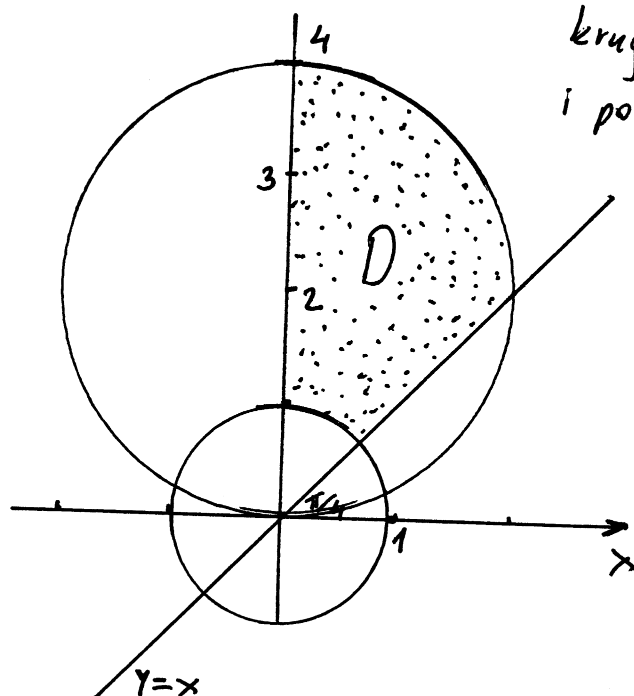
krug sa centrom $C(0;2)$ i poluprečnikom $r=2$

$x \le y$

$x=y$

$x \ge 0$

$x=0$



uvedimo polarne koordinate

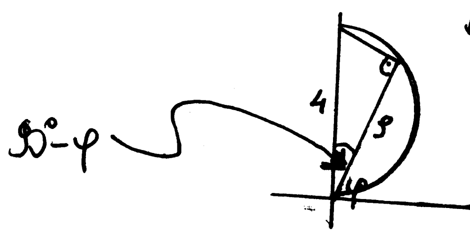
$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$dx\,dy = \rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transform.}} D'$

$D' : \begin{cases} \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \\ 1 \le \rho \le 4\sin\varphi \end{cases}$



$\underbrace{\cos(90^\circ-\varphi)}_{=\sin\varphi} = \frac{\rho}{4} \Rightarrow \sin\varphi = \frac{\rho}{4}$

$$I = \iint\limits_D x\,dx\,dy = \left| \begin{array}{l}\text{uvodimo}\\ \text{polarne}\\ \text{koordinate}\end{array} \right| = \iint\limits_{D'} \rho\cos\varphi\,\rho\,d\rho\,d\varphi = \int_{\pi/4}^{\pi/2} \cos\varphi \int_1^{4\sin\varphi} \rho^2\,d\rho$$

$$= \frac{1}{3}\int_{\pi/4}^{\pi/2} \cos\varphi \, \underbrace{\rho^3\Big|_1^{4\sin\varphi}}_{64\sin^3\varphi-1} d\varphi = \frac{64}{3}\int_{\pi/4}^{\pi/2} \sin^3\varphi\cos\varphi\,d\varphi - \frac{1}{3}\int_{\pi/4}^{\pi/2}\cos\varphi\,d\varphi = \ldots = \frac{\sqrt{2}}{6} + \frac{11}{3}$$

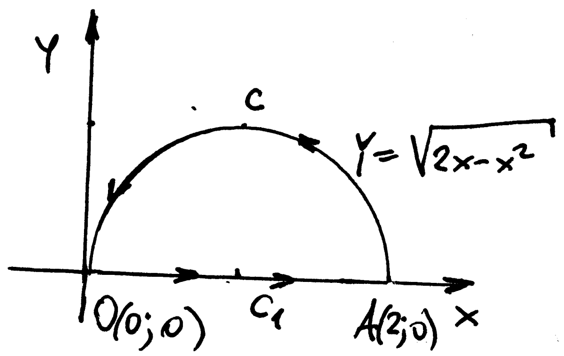
traženo rješenje

(#) Izračunati

$$I=\int_C \left(e^{x+y}\sin 2y+x+y\right)dx+\left(e^{x+y}(2\cos 2y+\sin 2y)+2x\right)dy$$

gdje je C kriva $y=\sqrt{2x-x^2}$, integracija se vrši od tačke $A(2;0)$ do tačke $O(0;0)$.

Rj. Skicirajmo krivu $y=\sqrt{2x-x^2}$.



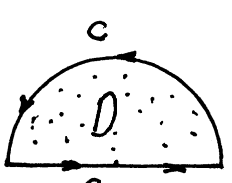
$$y^2=2x-x^2$$

$$x^2-2\cdot x\cdot 1+1+y^2=1$$

$$(x-1)^2+y^2=1$$

krug sa centrom u $C(1;0)$ polupr. $r=1$

Označimo sa C_1 krivu (pravu) od tačke $O(0;0)$ do $A(2;0)$, i označimo sa $P(x,y)=e^{x+y}\sin 2y+x+y$, $Q(x,y)=e^{x+y}(2\cos 2y+\sin 2y)+2x$. Neka je J integral u kome se integracija vrši prvo po krivoj C od tačke $A(2;0)$ do tačke $O(0;0)$ pa onda po krivoj C_1 od tačke $O(0;0)$ do tačke $A(2;0)$



$$J=\int_{CC_1} P(x,y)\,dx+Q(x,y)\,dy=\underbrace{\int_C P\,dx+Q\,dy}_{=I}+\int_{C_1} P\,dx+Q\,dy$$

$$\Rightarrow\quad I=J-\overbrace{\int_{C_1} P\,dx+Q\,dy}^{=J_1}$$

Zašto ovo?

Primjetimo da integral J možemo izračunati pomoću formule Greena

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= e^{x+y}(2\cos 2y + \sin 2y) + 2 \\ \frac{\partial P}{\partial y} &= e^{x+y}\sin 2y + e^{x+y} 2\cos 2y + 1 \end{aligned}\right\} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$J = \iint_D dx\,dy = \left| \begin{array}{l} \text{kako je } D \text{ polukrug} \\ \text{a znamo da se} \\ \text{površina kruga računa} \\ \text{po formuli } r^2\pi \end{array} \right| = \frac{1}{2}\pi$$

$$J_1 = \left| C_1 : \begin{cases} y = 0 \\ 0 \le x \le 2 \end{cases} \right| = \int_0^2 x\,dx = \frac{1}{2}x^2 \Big|_0^2 = 2$$

Prema tome

$$I = J - J_1 = \frac{\pi}{2} - 2$$ traženo rješenje

(#) Izračunati krivolinijski integral

$$I = \int_{\widehat{AO}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$

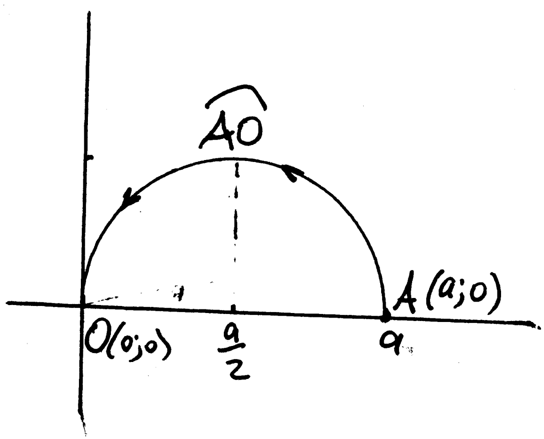
gdje je $\widehat{AO}$ gornji polukrug $x^2+y^2=ax$, $y \geq 0$ $(a>0)$ orjentisan od tačke $A(a;0)$ do tačke $O(0;0)$.

Rj. $x^2+y^2=ax$

$$x^2 - 2\cdot x\cdot \frac{a}{2} + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 = 0$$

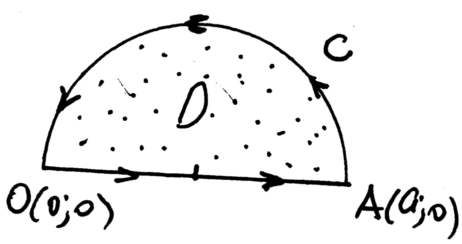
$$\left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

krug sa centrom u $C\left(\frac{a}{2}, 0\right)$
poluprečnika $r=\frac{a}{2}$



Primjetimo da kriva $\widehat{AO}$ nije zatvorena, pa ne možemo primjeniti formulu Greena. Međutim, ako zatvorimo polukrug $\widehat{AO}$ sa duži $\overline{OA}$ dobićemo integral

$$J = \int_{C} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$



na koji možemo upotrebiti formulu Greena. Sada primjetimo

$$J = I + \int_{\overline{OA}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy$$

tj.

$$I = J - \underbrace{\int_{\overline{OA}} (e^x \sin y - my)\,dx + (e^x \cos y - m)\,dy}_{=J_1}$$

Greenova formula

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Izračunajmo $J = \int_C \underbrace{(e^x \sin y - my)}_{P}\,dx + \underbrace{(e^x \cos y - m)}_{Q}\,dy$

primjenom Greenove formule

$$\frac{\partial P}{\partial y} = e^x \cos y - m \qquad \frac{\partial Q}{\partial x} = e^x \cos y$$

$$J = \iint_D m\,dx\,dy = m \underbrace{\iint_D dx\,dy}_{\substack{\text{površina polukruga} \\ P = r^2\pi}} = m \cdot \frac{1}{2}\left(\frac{a}{2}\right)^2 \pi = \frac{1}{8} a^2 m \pi$$

$$J_1 = \left| \overline{OA} : \begin{cases} y = 0 \\ 0 \le x \le a \end{cases} \right| = \int_0^a (e^x \cdot 0 - m \cdot 0)\,dx + (e^x \cdot 1 - m) \cdot 0 = 0$$

Prema tome $I = J - J_1 = \frac{1}{8} a^2 m \pi$ traženo rješenje

Prvo izračunati integral $\int_0^\infty e^{-x}\sin(\alpha x)\,dx$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha)=\int_0^\infty x e^{-x}\cos(\alpha x)\,dx .$$

Rj.

$$I(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx=\left|\begin{matrix} u=e^{-x} & dv=\sin\alpha x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x\end{matrix}\right|=$$

$$=-\frac{1}{\alpha}e^{-x}\cos\alpha x\Big|_0^\infty-\frac{1}{\alpha}\int_0^\infty e^{-x}\cos(\alpha x)\,dx=\left|\begin{matrix} u=e^{-x} & dv=\cos\alpha x\,dx \\ du=-e^{-x} & v=\frac{1}{\alpha}\sin\alpha x\end{matrix}\right|$$

ovdje podrazumijevano da se računa $\lim\limits_{A\to\infty}(e^{-x}\cos\alpha x)\Big|_0^A$

$$=\left(0+\frac{1}{\alpha}\cdot 1\right)-\underbrace{\frac{1}{\alpha^2}e^{-x}\sin\alpha x\Big|_0^\infty}_{0-0}-\frac{1}{\alpha^2}\int_0^\infty e^{-x}\sin\alpha x\,dx$$

ovdje podrazumijevam da se računa $\lim\limits_{A\to\infty}(e^{-x}\sin\alpha x)\Big|_0^A$

$$\Rightarrow\quad I(\alpha)=\frac{1}{\alpha}-\frac{1}{\alpha^2}I(\alpha)$$

$$I(\alpha)+\frac{1}{\alpha^2}I(\alpha)=\frac{1}{\alpha}\quad\Rightarrow\quad\left(1+\frac{1}{\alpha^2}\right)I(\alpha)=\frac{1}{\alpha}$$

$$\frac{\alpha^2+1}{\alpha^2}I(\alpha)=\frac{1}{\alpha}\quad /\cdot\alpha$$

$$I(\alpha)=\frac{\alpha}{\alpha^2+1}$$

Označimo sa $F(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx=\frac{\alpha}{\alpha^2+1}$.

Kako je $\left(e^{-x}\sin(\alpha x)\right)'_\alpha=xe^{-x}\cos\alpha x$ i

$$\left(\frac{\alpha}{\alpha^2+1}\right)'_\alpha = \frac{1\cdot(\alpha^2+1)-\alpha\cdot 2\alpha}{(\alpha^2+1)^2} = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

To je

$$F'(\alpha) = \int_0^\infty x e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

Pa je

$$G(\alpha) = \int_0^\infty x e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

traženo rješenje

(#) Date su vrijednosti dva integrala ($\alpha > 0$)

$$\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx = \frac{\pi}{2}e^{-\alpha} \quad i \quad \int_0^\infty \frac{\sin\alpha x}{x}\,dx = \frac{\pi}{2}.$$

Koristeći date jednakosti uz pomoć metode diferenciranja po parametru izračunati

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx.$$

Rj: Označimo sa $F(\alpha) = \int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx = \frac{\pi}{2}e^{-\alpha}$.

Kako je $\left(\frac{\cos\alpha x}{1+x^2}\right)'_\alpha = \frac{-x\sin\alpha x}{1+x^2}$ i $\left(\frac{\pi}{2}e^{-\alpha}\right)'_\alpha = -\frac{\pi}{2}e^{-\alpha}$

To je $F'_\alpha = \int_0^\infty \frac{-x\sin\alpha x}{1+x^2}\,dx = -\frac{\pi}{2}e^{-\alpha}$ tj. $\int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx = \frac{\pi}{2}e^{-\alpha}$

Sad primjetimo da je

$$\frac{x\sin\alpha x}{1+x^2} = \frac{x^2\sin\alpha x}{x(1+x^2)} \overset{+\sin\alpha x - \sin\alpha x}{=} \frac{x^2\sin\alpha x + \sin\alpha x}{x(1+x^2)} - \frac{\sin\alpha x}{x(1+x^2)} =$$

$$= \frac{\sin\alpha x\cdot(x^2+1)}{x\cdot(1+x^2)} - \frac{\sin\alpha x}{x(1+x^2)} = \frac{\sin\alpha x}{x} - \frac{\sin\alpha x}{x(1+x^2)}$$

Pa imamo

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx = \overbrace{\int_0^\infty \frac{\sin\alpha x}{x}\,dx}^{\frac{\pi}{2}} - \overbrace{\int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx}^{\frac{\pi}{2}e^{-\alpha}} = \frac{\pi}{2}\left(1-e^{-\alpha}\right)$$